18.152 PROBLEM SET 2 SOLUTIONS

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1. Problem 1

Most students realized the relation of this problem with Theorem 2 in Lecture Note 2. One way to proceed is to generalize Theorem 2 first to higher dimensions; let us state the theorem:

Theorem 1.1. *Let Ω be a bounded convex open set of $\mathbb{R}^n$. Suppose that a smooth function u satisfies*

$$\begin{cases} u_t & = \Delta u \text{ in } \overline{Q}_T \text{ where } Q_T = \Omega \times [0, T), \\ u(x, 0) & = g(x) \text{ for } x \in \overline{\Omega} \text{ and some } g : \overline{\Omega} \to \mathbb{R} \text{ smooth}, \\ u(\sigma, t) & = 0 \text{ for } \sigma \in \partial\Omega, t \in [0, \Omega). \end{cases}$$

Then the following holds in $\overline{Q}_T$:

$$\|\nabla u(x, t)\| \leqslant \max_{x \in \overline{\Omega}} \|\nabla g(x)\|.$$

Solution to Problem 1. The main problem here is that the function $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ does not satisfies the boundary condition along $\partial\Omega \times [0, T]$ in Theorem 1.1, so it is necessary to make a correction and apply Theorem 1.1 to a different function.

Consider the function

$$w_1 : \overline{\Omega} \times [0, T] \to \mathbb{R},$$
$$(x, t) \mapsto w(x),$$

so w_1 is time-independent and solves the heat equation:

$$(w_1)_t = \Delta w_1,$$

since $w : \overline{\Omega} \to \mathbb{R}$ is harmonic. Now the difference

$$u_1 := u - w_1 : \overline{\Omega} \times [0, T] \to \mathbb{R},$$

satisfies all conditions in Theorem 1.1 with initial value:

$$u_1(x, 0) = u(x, 0) - w_1(x, 0) = g(x) - w(x).$$

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Then Theorem 1.1 implies that

$$\|\nabla(u-w_1)(x,t)\| = \|\nabla u_1(x,t)\| \leqslant \max_{x\in\overline{\Omega}} \|\nabla(g-w)(x)\|,$$

so for any $(x,t)\in\overline{Q}_T$,

$$\begin{aligned}\|\nabla u(x,t)\| &\leqslant \|\nabla(u-w_1)(x,t)\| + \|\nabla w_1(x,t)\| \\ &\leqslant \max_{x\in\overline{\Omega}}\|\nabla g\| + 2\max_{x\in\overline{\Omega}}\|\nabla w\|.\end{aligned}$$

Now we come to estimate the absolute value of u. Let $K=\max_{x\in\overline{\Omega}}|g(x)|$ and consider the function

$$u' = u - K.$$

Then u' satisfies the following properties:

$$\begin{cases} u'_t &= \Delta u' \text{ in } \overline{Q}_T, \\ u(x,0) &\leqslant 0 \text{ for } x\in\overline{\Omega} \\ u(\sigma,t) &\leqslant 0 \text{ for any } \sigma\in\partial\Omega, t\in[0,\Omega). \end{cases}$$

By the maximum principle, $u'\leqslant 0$ on $\overline{Q}_T$, so

$$u \leqslant K.$$

Apply the same argument for $-(u+K)$, and we obtain that

$$-K \leqslant u.$$

As a result, $|u(x,t)|\leqslant K$ for any $(x,t)\in\overline{Q}_T$. □

The proof of Theorem 1.1 follows the same line of argument in Lecture Note 2 using Barriers and is omitted here. The grader would like to encourage students to go through the proof and see how the boundary condition

$$u(\sigma,t) = 0 \text{ for } \sigma\in\partial\Omega, t\in[0,\Omega),$$

is used in the proof of Theorem 1.1.

2. Problem 4

Only two students figured out a complete solution to Problem 4. Most of students got the right formula, but they didn't realize the difference between the Hessian and Laplacian:

$$|\nabla^2 v|^2 := \sum_{1\leqslant i,j\leqslant n} |\partial_i\partial_j v|^2 \text{ and}$$

$$|\Delta v|^2 := (\sum_{i=1}^{n} \partial_i^2 v)^2.$$

They are the same only if the dimension is 1.

Solution to Problem 4. We follow the proof of Theorem 5 in Lecture Note 2. Let $v = \log u$, then

$$\begin{aligned}\partial_t v &= \frac{\partial_t u}{u},\\ \partial_i v &= \frac{\partial_i u}{u},\\ \partial_i^2 v &= \frac{\partial_i^2 u}{u} - (\frac{\partial_i u}{u})^2, 1 \leqslant i \leqslant n,\end{aligned}$$

so

$$\begin{aligned}-\partial_t v + \Delta v + |\nabla v|^2 &= -\frac{\partial_t u}{u} + \sum_{i=1}^n (\partial_i^2 v + |\partial_i v|^2)\\ &= \frac{-\partial_t u + \Delta u}{u} = 0.\end{aligned}$$

Apply the Laplacian operator Δ to the equation above:

$$\begin{aligned}\partial_t(\Delta v) &= \Delta(\Delta v) + \Delta|\nabla v|^2\\ &= \Delta(\Delta v) + \sum_{i=1}^n \partial_i^2 \sum_{j=1} |\partial_j v|^2\\ &= \Delta(\Delta v) + 2\sum_{i=1}^n \sum_{j=1}^n \partial_i \langle \partial_i \partial_j v, \partial_j v\rangle\\ &= \Delta(\Delta v) + 2\sum_{i=1}^n \sum_{j=1}^n \langle \partial_i^2 \partial_j v, \partial_j v\rangle + |\partial_i \partial_j v|^2,\\ &= \Delta(\Delta v) + 2\sum_{j=1}^n \langle \partial_j \Delta v, \partial_j v\rangle + 2\sum_{1\leqslant i,j\leqslant n} |\partial_i \partial_j v|^2\\ &= \Delta(\Delta v) + 2\langle \nabla\Delta v, \nabla v\rangle + 2|\nabla^2 v|^2.\end{aligned}$$

For any $\epsilon > 0$ and $S > 0$, consider the function

$$w := \Delta v + \frac{\frac{n}{2} + \epsilon}{t + S}$$

defined when $t > -S$, then

$$\partial_t w = \partial_t \Delta v - \frac{\frac{n}{2} + \epsilon}{(t+S)^2},\ \nabla w = \nabla\Delta v, \Delta w = \Delta(\Delta v).$$

so

$$\partial_t w = \Delta w + 2\langle \nabla w, \nabla v\rangle + 2|\nabla^2 v|^2 - \frac{\frac{n}{2} + \epsilon}{(t+S)^2}. \tag{1}$$

We claim that $w \geqslant 0$ for all $t > -S$. Since w is periodic and $\lim_{t\to -S} w(x) = \infty$ holds for all $x \in \mathbb{R}^n$, if $w < 0$ at some point then there exists some space-time point $(x_0, t_0) \in \mathbb{R}^n \times (-S, T)$ such that $w(x_0, t_0) = 0$ and $w(x, t) > 0$ for all $x \in \mathbb{R}^n$ and $t \in (-S, t_0)$. Therefore, at the time slice $\mathbb{R}^n \times \{t_0\}$,

$$w(x_0, t_0) = \min_{x\in\mathbb{R}_n} w(x_, t_0).$$

As a result,

$$0 \geqslant \partial_t w(x_0, t_0), \nabla w(x_0, t_0) = 0, \Delta w(x_0, t_0) \geqslant 0,$$

and at (x_0, t_0), the equation (1) yields:

$$0 \geqslant 2|\nabla^2 v|^2 - \frac{\frac{n}{2} + \epsilon}{(t + S)^2}. \tag{2}$$

On the other hand, $w(x_0, t_0) = 0$ implies that

$$\Delta v(x_0, t_0) = -\frac{\frac{n}{2} + \epsilon}{t + S}. \tag{3}$$

To draw a contraction from (2) and (3), we use the Cauchy-Schwartz inequality:

$$\begin{aligned} |\Delta v(x_0, t_0)|^2 = |\sum_{1\leqslant i\leqslant n} \partial_i^2 v|^2 &\leqslant n \sum_{1\leqslant i\leqslant n} |\partial_i^2 v|^2 \\ &\leqslant n \sum_{1\leqslant i,j\leqslant n} |\partial_i \partial_j v|^2 = n|\nabla^2 v|^2. \end{aligned}$$

We obtain a contradiction by plugging in (2) and (3):

$$\left(\frac{\frac{n}{2} + \epsilon}{t + S}\right)^2 \leqslant \frac{n}{2} \cdot \frac{\frac{n}{2} + \epsilon}{(t + S)^2},$$

if $\epsilon > 0$. As a result, $w \geqslant 0$ and

$$\Delta v \geqslant -\frac{\frac{n}{2} + \epsilon}{t + S}.$$

By taking $S \to \infty$, we obtain that $\Delta v \geqslant 0$ and

$$\frac{\partial_t u}{u} = \partial_t v = \Delta v + |\nabla v|^2 \geqslant 0.$$

□